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SOME NOTES ABOUT AFFINE DIAMETERS OF CONVEX FIGURES

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Given a point x in a convex figure M, let $\gamma(x)$ denote the number of all affine diameters of M passing through x. It is shown that, for a convex figure M, the following conditions are equivalent.

(i) $\gamma(x) \geq 2$ for every point $x \in \operatorname{int} M$.

(ii) either $\gamma(x) \equiv 3$ or $\gamma(x) \equiv \infty$ on $\operatorname{int} M$. Furthermore, the set $B = \{x \in \operatorname{int} M : \gamma(x)$ is either odd or infinite $\}$ is dense in M.

Let M be a convex figure, i.e., a closed convex set in the plane with nonempty interior. A chord $[a, b]$ of M is said to be an *(affine) diameter* of M if and only there exists a pair of different parellel supporting lines of M each containing one of the endpoints a, b. For some properties of diameters and their generalizations see for example [1], [3].

Denote by $\gamma(x)$ the number of all diameters of M passing through a point $x \in M$. Let $\gamma(x) = \infty$ if infinitely many diameters pass through x. It was mentioned by Hammer [4] that each point $x \in M$ belongs to at least one diameter. In other words, $\gamma(x) \geq 1$.

The following result of Eggleston is well-known.

Theorem ([5]). *A convex figure M is a triangle if and only if every point of its interior belongs to exactly three diameters, i.e., $\gamma(x) \equiv 3$ on* $\operatorname{int} M$.

This result is a particular case of

Theorem 1. *For any convex figure M, the following two conditions are equivalent.*
(i) M is a triangle,
(ii) $2 \leq \gamma(x) < \infty$ *for every point* $x \in \operatorname{int} M$.

This can be deduced from Eggleston's theorem and the following result.

Theorem 2. *For any convex figure M, the following conditions are equivalent.*
(i) $\gamma(x) \geq 2$ *for every point* $x \in \operatorname{int} M$,
(ii) either $\gamma(x) \equiv 3$ *or* $\gamma(x) \equiv \infty$ *on* $\operatorname{int} M$.

For the proof of Theorem 2 we need two lemmas.

Lemma 3. *If the diameters $[a, b]$ and $[c, d]$ of M have no point in common, and a, b, c, d are four points of the boundary of M (in this order), then the arcs $\overset{\frown}{ad}$ and $\overset{\frown}{bc}$ are parallel line segments.* ∎

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Lemma 4. *Let the diameters $[a, b]$ and $[c, d]$ of M have a common point in $\operatorname{int} M$, and let x be any point in the region bounded by these diameters and the arcs $\widehat{ac}, \widehat{bd}$. Then there exists a diameter of M that passes through x, and its endpoints are on the arc $\widehat{ac}$ and $\widehat{bd}$, respectively.*

Proof. Suppose that x is in the interior of the region bounded by the diameters $[a, b], [c, d]$ and the arc $\widehat{ac}$. If the rays $\overrightarrow{b,x}$ and $\overrightarrow{d,x}$ intersect $\operatorname{bd} M$ in points b' and d', respectively, then $\widehat{b'd'} \subset \widehat{ac}$. Denote by $[e_1, f_1]$ a chord which minimizes the value $\|e - x\|/\|x - f\|$ over all chords $[e, f], e \in \widehat{b'd'}$, $f \in \widehat{bd}$ passing through x. We claim that $[e_1 f_1]$ is a diameter of M.

Suppose the contrary and denote by ℓ_1, ℓ_2 the two different parallel supporting lines of M, where $e_1 \in \ell_1$. If $f_2 \in M \cap \ell_2$, then $f_2 \in \widehat{bd}$ and $f_2 \neq f_1$, by assumption. Let f_1' be the point of intersection of the lines ℓ_2 and (e_1, f_1), and let e_2 be the point of intersection of $\overrightarrow{f_2,x}$ with $\operatorname{bd} M$. Then

$$\frac{\|e_2 - x\|}{\|x - f_2\|} \leq \frac{\|e_1 - x\|}{\|x - f_1'\|} < \frac{\|e_1 - x\|}{\|x - f_1\|},$$

contradicting the choice of $[e_1, f_1]$. ∎

Proof of Theorem 2. Suppose first that there is a diameter $[y, z]$ of M lying completely in $\operatorname{bd} M$. Denote by ℓ the supporting line of M parallel to $[y, z]$. Let $M \cap \ell = [v, w]$. If it is possible to pick a point p in the set $\operatorname{int} M \setminus \operatorname{conv}([y, z] \cup [v, w])$, sufficiently close to v, then each diameter of M passing through p is of the form $[t, z]$. Therefore $\gamma(p) = 1$. Because of the convexity of M, we obtain that $[v, y] \subset \operatorname{bd} M$. Similarly $[w, z] \subset \operatorname{bd} M$. Thus, M is either a trapezoid or a triangle (if $v = w$). In the first case $\gamma(x) \equiv \infty$ on $\operatorname{int} M$, and in the second case $\gamma(x) \equiv 3$ on $\operatorname{int} M$.

Assume next that $\operatorname{bd} M$ does not contain any diameter of M. We are going to show that in this case $\gamma(x) \equiv \infty$ on $\operatorname{int} M$. Supppose, in order to obtain a contradiction, that $\gamma(x_0) < \infty$ for some point $x_0 \in \operatorname{int} M$. We can find a point $v \in \operatorname{int} M$ and two diameters $[a, b], [c, d]$ through v, not containing x_0, otherwise $\gamma(x_0) = \infty$. Assume without loss of generality that x_0 is in the region bounded by the diameters $[a, b], [c, d]$ and the arc $\widehat{ac}$. There is a diameter $[y, z]$ passing through x_0, for which $y \in \widehat{ac}, z \in \widehat{bd}$. Obviously $a \neq y \neq c$. Since $\gamma(t) \geq 2$, there exists a diameter $[v, w]$ different from $[y, z]$, passing through a point t. We claim that $v \in \operatorname{int} \widehat{ac}$. Indeed, otherwise we could choose a sequence of points $t_1, t_2, \ldots$ from (x_0, y) tending to y so that every diameter $[\alpha_n, \beta_n]$ of M containing t_n and different from $[y, z]$ statisfies the condition $\alpha_n, \beta_n \notin \operatorname{int} \widehat{ac}$. Let $[\alpha, \beta]$ be a limiting chord of some subsequence of the sequence $[\alpha_n, \beta_n]$, $n \geq 1$. It is clear that $y \in (\alpha, \beta)$ and $[\alpha, \beta]$ is a diameter of M. Therefore $[\alpha, \beta] \subset \operatorname{bd} M$, which is impossible by our assumption.

So, if t is sufficiently close to y, then $v \in \operatorname{int} \widehat{ac}$. If $w \notin \widehat{bd}$, then diameter $[v, w]$ has no point in common with one of the diameters $[a, b], [c, d]$. Suppose, say, that $[v, w] \cap [a, b] = \emptyset$. By Lemma 1, we have that $\widehat{av} = [a, v] \,|\, [bw] = \widehat{bw}$. Now $x_0 \in \operatorname{intconv}([a, b] \cup [v, w])$ implies $\gamma(x_0) = \infty$. Hence, $w \in \widehat{bd}$.

Assume that $w = d$ (the case $w = b$ can be treated similarly). Denote by e the intersection point of the ray $\overrightarrow{d, x_0}$ and $\operatorname{bd} M$. It is easy to see that $[d, e]$ is a diameter of M. If a point $s \in (x_0, e)$ is sufficiently close to e then, according to the argument demonstrated above , some other diameter $[p,q]$ passes through s so that $p \in \operatorname{int} \widehat{cv}$. Diameter $[p,q]$ has no point in common with one of diameters $[c,d],[v,w]$. As above, we obtain $\gamma(x_0)=\infty$.

Hence, we can suppose that $w \in \operatorname{int} \widehat{bd}$. Suppose e.g. that $w \in \widehat{dz}$. Repeating the above arguments we obtain another diameter $[y',x'], y' \in \operatorname{int} \widehat{cv}, z' \in \operatorname{int} \widehat{dw}$ passing through x_0. Evidently, $[y,z]$ and $[y'z']$ are different.

Proceeding in the same way, we will find an infinite sequence of different diameters countaining x_0, i.e., $\gamma(x_0)=\infty$, a contradiction. ∎

Corollary 5. *If $\gamma(x) \neq$ const on $\operatorname{int} M$, then there is a point $x_0 \in \operatorname{int} M$ with $\gamma(x_0)=1$.*

In view of Theorem 6 and Corollary 5, it might be of some interest to learn more about the values of the function $\gamma(x)$, and about its level sets. A result in this direction is the following.

Theorem 6. *For any convex figure M, the set*

$$B=\{x \in \operatorname{int} M : \gamma(x) \quad \text{is either odd or infinite}\}$$

is dense in M.

For the proof of Theorem 6 we need some lemmas. We will say that a line ℓ *almost lies* in a region P, if only some bounded part of ℓ does not belong to P.

Let $a_1,\dots,a_{2n}$ be a family of lines in the plane passing through a point O, and indexed counterclockwise. Denote by P_i the double–cone bounded by a_i and a_{i+1} (in this clockwise order), $i=1,\dots,2n-1$. Let P_{2n} be the double–cone bounded by a_{2n} and a_1. Then O cuts each line a_i in two rays ℓ_i', ℓ_i''. Let $b_1,\dots,b_{2n}$ be some lines, not containing O, so that b_i almost lies in $P_i, i=1,\dots,2n$. Denote by δ_i' (δ_i'') the number of points of intersection of ℓ_i' (ℓ_i'') with $b_1,\dots,b_{2n}$, and put $\delta_i=|\delta_i'-\delta_i''|$.

Lemma 7. *At least one of the numbers $\delta_1,\dots,\delta_{2n}$ is not equal to zero.*

Proof. We will establish Lemma 7 by induction on n. The case $n=1$ is trivial. Suppose, that our assertion is true for all $n \le m-1$. The lines $a_i, b_i, i=1,\dots 2m$ will be considered in the cyclic order
$b_{2m}a_1b_1a_2b_2\dots a_{2m}b_{2m}$. We say that b_{i-1} and b_i are the *neighbours* of a_i.

We claim that some line a_i is intersected by its neighbours on the same side relative to 0. Assume the contrary, and direct the lines $a_1,\dots,a_{2n}$ arbitrarily. Suppose without loss of generality that b_1 intersects a_1 on its positive side. Then all of the lines $a_2,a_4,\dots,a_{2m}$ are intersected by their neighbours on their negative sides. Therefore b_{2m} intersects a_1 on its positive side, i.e., on the same side where b_1 does, contradiction.

Assume without loss of generality that a_2 is intersected by its neighbours b_1, b_2 at the same side relative to O. Let $\overline{a}_1,\overline{a}_2,\dots,\overline{a}_{2m-2}$ and $\overline{b}_1,\overline{b}_2,\dots \overline{b}_{2m-2}$ denote the lines $a_1,a_4,a_5,\dots,a_{2m}$ and $b_3,b_4,\dots,b_{2m-2}$, respectively. By our induction hypothesis, one of the numbers $\overline{\delta}_1,\dots,\overline{\delta}_{2m-2}$ for this family of lines is not equal to zero. It is easy to see that b_1 and b_2 intersect each line $a_i, i \neq 2$ on different sides relative to O. Therefore,

$$\overline{\delta}_1=\delta_1, \overline{\delta}_2=\delta_4, \overline{\delta}_3=\delta_5,\dots,\overline{\delta}_{2m-2}=\delta_{2m}\,.$$

Hence, one of the numbers $\delta_1,\dots,\delta_{2m}$ is not equal to zero. ∎

Lemma 8. *Suppose that $\gamma(x)$ is finite for some point $x \in \operatorname{int} M$. Then*
(i) there is a neighbourhood $U \subset \operatorname{int} M$ of x such that $\gamma(y)+\gamma(z) > 2\gamma(x)$ for any pair of points $y, z \in U$ with $x \in (y,z)$;
(ii) there is a closed set $V \subset \operatorname{int} M$ having nonempty interior such that $x \in \operatorname{bd} V$, and
$\gamma(z) > \gamma(x)$ for any $z \in V \setminus \{x\}$ if $\gamma(x)$ is even,
$\gamma(z) \geq \gamma(x)$ for any $z \in V$ if $\gamma(x)$ is odd.

Proof. Let $\gamma(x) = n$, and $a_1, \dots a_n$ be all the diameters of M passing through x. These diameters divide M into n closed regions $P_1, \dots, P_n$, where P_i is the part of the double–cone with apex x, bounded by the diameters a_i, a_{i+1} and $\operatorname{bd} M$. (Similarly, P_n is bounded by the diameters a_n, a_1 and $\operatorname{bd} M$.) According to Lemma 4 it is possible to find n new diameters $b_1, \dots, b_n$ such that both endpoints of b_i are in P_i. Using the condition $\gamma(x) = n$, we obtain that none of the diameters $b_1, \dots b_n$ contain x. Therefore, one can find a closed disc $\Sigma_\tau(x)$ of radius $\tau > O$ and center x, disjoint from the chords $b_1, \dots, b_n$. Let $v_{1i}, \dots, v_{ni}$, $i = 1, \dots, n$, denote the points of intersection of b_i with $a_1, \dots, a_n$, respectively. (Some of these points may coincide with each other or with some endpoints of the diameters b_i.)

(i) We will show that the neighbourhood $U = \Sigma_\tau(x)$ meets the requirements of the lemma. Let $y, z \in U$ satisfy the condition $x \in (y,z)$, and let ℓ denote the line determined by y and z. Suppose that the chord $\ell \cap M$ lies almost in P_1 (say). We distinguish two cases, according to whether ℓ contains one of the chords a_1 and a_2, or not.

Case 1. Let e.g. $\ell \cap M = a_1$. If (say) y belongs to the segment $[x, v_{1i}], i = 2, \dots, n-1$, then applying Lemma 4 to the pairs of chords a_i, b_i and b_i, a_{i+1} we obtain the existence of two diameters, different from $a_1, \dots, a_n$ and passing through y such that their endpoints lie in P_i. Furthermore, if $[x, v_{11}] \cap [x, v_{1n}]$ contains one of the points y, z, then there are at least three diameters through this point, ending in P_1 or P_n (one of these diameters is a_1). Clearly, at least one diameter (namely a_1) passes through the other point. If both $[x, v_{11}]$ and $[x, v_{1n}]$ contain at least one of the points y, z, then there are at least two diameters through y and z each, ending in P_1 or P_n. Hence,

$$\gamma(y) + \gamma(z) \geq 2(n-2) + 4 = 2\gamma(x).$$

Case 2. Assume that ℓ does not contain any line a_1, a_2, and let $w_2, \dots, w_n$ be the points of intersection of ℓ with $b_2, \dots, b_n$, respectively. Similarly to the previous case, if the interval $[x, w_i]$ contains y (or z), then there are at least two diameters throught this point, different from $a_1, \dots, a_n$, such that they end in P_i. In addition, there are two diameters ending in P_1, passing through y and z, respectively. Hence, again we obtain

$$\gamma(y) + \gamma(z) \geq 2(n-1) + 2 = 2\gamma(x),$$

as required.

(ii) Let $\gamma(x) = n = 2k$. By Lemma 7, some diameter $a_i, i = 1, \dots, 2k$, contains more than k points from the set $\{v_{i1}, \dots, v_{in}\}$ at the same side relative to x. Without loss of generality we will assume that $i = 1$ and that the ray $\ell_1 = \overrightarrow{x, a_1}$ (pointing in the positive direction of a_1) contains $m \geq k+1$ points $v_{1i_p}, p = 1, \dots, m$. Let ℓ_2 be a ray in P_1 sufficiently close to ℓ_1 so that ℓ_2 intersects all diameters $b_{i_p}, p = 1, \dots, m; i_p \neq 1$.

Denote by V the piece of $\Sigma_T(x)$ between the rays ℓ_1, ℓ_2. Let $z \in V \setminus \{x\}$. There are two possibilities.

Case 1. $\overrightarrow{x,z} = \ell_1$. Then z belongs to all of $[x, v_{1i_p}], p = 1, \dots, m$. Analogously to Case 1 part (i), we get

$$\gamma(z) \geq 2(m-2) + 3 \geq 2k + 1 > \gamma(x).$$

Case 2. $\overrightarrow{x,z} \neq \ell_1$. The endpoints of the segment $(x, z) \cap M$ belong to P_1. Therefore this segment intersects all the diameters $b_{i_p}, i = 1, \dots, m; i_p \neq 1$ in some points z_{i_p}. Each interval $[x, z_{i_p}]$ contains z. Now we obtain, as before, that

$$\gamma(z) \geq 2(m-1) + 1 \geq 2k + 1 > \gamma(x).$$

Let $\gamma(x) = n = 2k + 1$. One of the two sides of the diameter a_1 with respect to x contains at least $m \geq k + 1$ points of the type v_{1i}. As above, choose a ray $\ell_2 \subset P_1$ and a neighbourhood V. If $z \in \ell_1$, then

$$\gamma(z) \geq 2(m-2) + 3 \geq 2k + 1 = \gamma(x).$$

If $z \notin \ell_1$, then

$$\gamma(z) \geq 2(m-1) + 1 \geq 2k + 1 = \gamma(x). \qquad \blacksquare$$

Proof of Theorem 6. Assume in order to obtain a contradiction that there is an open neighbourhood U of some point $x_0 \in \operatorname{int} M$ such that $\gamma(x)$ is even (and finite) for all $z \in U$. According to Lemma 4, there exists a closed subset $V_1 \subset U$ with nonempty interior so that $x_0 \in \operatorname{bd} V_1$ and $\gamma(z) > \gamma(x_0)$ for every $z \in V_1 \setminus \{x_0\}$. Choose an arbitrary point $x_1 \in \operatorname{int} V_1$. By Lemma 4, there exists a closed subset $V_2 \subset \operatorname{int} V_1$ with nonempty interior such that $x_1 \in \operatorname{bd} V_2$ and $\gamma(z) > \gamma(x_1)$ for every $z \in V_2 \setminus \{x_1\}$.

Proceeding like this, we construct a sequence $V_1 \supset V_2 \supset \dots$ of compact sets. Obviously, the set $V = \cap V_i$ is nonempty and $\gamma(z) = \infty$ for every $z \in V$. The last fact is in contradiction with the choice of U. Hence $\overline{B} = M$, completing the proof. $\blacksquare$

References

[1] B. GRÜNBAUM: Measures of symmetry for convex sets, In V. Klee ed., Proceedings Symp. Pure Math., Vol **VII.**, *Amer. Math. Soc.*, **(1963)**, 233–270.

[2] B. GRÜNBAUM: Continuous families of curves, *Canad. J. Math.*, **18** (1966) 529–537.

[3] T. ZAMFIRESCU: On continuous families of curves. *VI. Geom. Dedic.*, **10** (1981) 205–217.

[4] P. C. HAMMER: Diameters of convex bodies, *Proc. Amer. Math. Soc.*, **5** (1954) 304–306.

[5] H. G. EGGLESTON: Some properties of triangles as external convex curves, *J. London Math. Soc.*, **28** (1953) 32–36.

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